4.10.1 Frequency Response of a System

$$F_{\text{ext}}(t) = F_o \sin(\omega t) \tag{4.76}$$

$$x(t) = X_o \sin(\omega t + \phi) \tag{4.77}$$

Analytical Procedure to Determine the Frequency Response of a System

1. *Find the* ***Laplace transform*** *of the system differential equation assuming initial conditions are zero:* $x(0) = \mathrm{d}x / \mathrm{d}t\,(0) = 0$.

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + 2\zeta\omega_n \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \omega_n^{\;2} x(t) = \frac{\omega_n^{\;2}}{k} F_{\text{ext}}(t) \tag{4.78}$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^{\;2}) X(s) = \frac{\omega_n^{\;2}}{k} F_{\text{ext}}(s) \tag{4.79}$$

2. *Find the* **transfer function** *of the system, which is the ratio of the output and input Laplace transforms.*

$$G(s) = \frac{X(s)}{F_{\text{ext}}(s)} = \frac{\dfrac{\omega_n^{\;2}}{k}}{(s^2 + 2\zeta\omega_n s + \omega_n^{\;2})} \tag{4.80}$$

3. *To simulate a harmonic input, replace s with* $\mathrm{j}\omega$ *in the transfer function. This yields the frequency response behavior of the system.*

$$G(\mathrm{j}\omega) = \frac{1/k}{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right] + \mathrm{j}\left(2\zeta\dfrac{\omega}{\omega_n}\right)} \tag{4.81}$$

4. *Find the desired amplitude ratio between the output and input by determining the magnitude of the complex transfer function:*

$$\text{mag}[G(\mathrm{j}\omega)] = |G(\mathrm{j}\omega)| \tag{4.82}$$

$$\frac{X_o}{F_o/k} = \frac{1}{\left\{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\dfrac{\omega}{\omega_n}\right)^2\right\}^{1/2}} \tag{4.83}$$

5. *Find the phase angle* ϕ *between the output and input by determining the argument of the complex transfer function:*

$$\phi = \arg[G(\mathrm{j}\omega)] = \angle G(\mathrm{j}\omega) \tag{4.84}$$

$$\phi = 0 - \tan^{-1}\left\{\frac{2\zeta\dfrac{\omega}{\omega_n}}{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]}\right\} = -\tan^{-1}\left\{\frac{2\zeta}{\dfrac{\omega_n}{\omega} - \dfrac{\omega}{\omega_n}}\right\} \tag{4.85}$$