# Chapter 4 Summary
# System Response

high fidelity measurement system requirements:

1. Amplitude linearity
2. Adequate bandwidth
3. Phase linearity

*amplitude linearity:* $V_{\text{out}}(t) - V_{\text{out}}(0) = \alpha[V_{\text{in}}(t) - V_{\text{in}}(0)]$

*Fourier series:* $F(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$

*Fourier coefficients:*

$$A_n = \frac{2}{T}\int_0^T F(t)\cos(n\omega_0 t)dt$$

$$B_n = \frac{2}{T}\int_0^T F(t)\sin(n\omega_0 t)dt$$

$$C_0 = \frac{1}{T}\int_0^T F(t)dt = \frac{A_0}{2}$$

*decibel:* $$\text{dB} = 20\log_{10}\left(\frac{A_{\text{out}}}{A_{\text{in}}}\right)$$

***frequency response curve***: plot of the ***amplitude ratio***, $A_{\text{out}}/A_{\text{in}}$, vs. the input frequency

***bandwidth***: the range of frequencies a system can adequately reproduce.

*cutoff frequency points:* $$\frac{A_{\text{out}}}{A_{\text{in}}} = \sqrt{\frac{P_{\text{out}}}{P_{\text{in}}}} = \sqrt{\frac{1}{2}} \approx 0.707$$

*phase linearity:* $$f = k \cdot f$$

zero order system:

$$A_0 X_{\text{out}} = B_0 X_{\text{in}}$$

$$X_{\text{out}} = \frac{B_0}{A_0} X_{\text{in}} = K X_{\text{in}}$$

first order system:

$$\tau \frac{dX_{\text{out}}}{dt} + X_{\text{out}} = KX_{\text{in}}$$

$$X_{\text{out}}(t) = KA_{\text{in}}(1 - e^{-t/\tau})$$

second order system:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_{\text{ext}}(t)$$

*undamped (b = 0):* $x_h(t) = A\cos\left(\sqrt{\frac{k}{m}}\ \text{t}\right) + B\sin\left(\sqrt{\frac{k}{m}}\ \text{t}\right)$

*natural frequency:* $\omega_n = \sqrt{\frac{k}{m}}$

*critically damped (b = $b_c$):* $x_h(t) = (A + Bt)e^{-\omega_n t}$

*critical damping constant:* $b_c = 2\sqrt{km} = 2m\omega_n$

*damping ratio:* $\zeta = \frac{b}{b_c} = \frac{b}{2\sqrt{km}}$

*underdamped ($\zeta$ < 1):*

$$x_h(t) = e^{-\zeta\omega_n t}[A\cos(\omega_n\sqrt{1-\zeta^2}\ \text{t}) + B\sin(\omega_n\sqrt{1-\zeta^2}\ t)]$$

*damped natural frequency:* $\omega_d = \omega_n\sqrt{1-\zeta^2}$

*overdamped ($\zeta$ > 1):*

$$x_h(t) = Ae^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + Be^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

frequency response analysis:

*input:* $x_{in}(t) = A_{in}\sin(\omega t)$

*output:* $x_{out}(t) = A_{out}\sin(\omega t + \phi)$

*transfer function:* $G(s) = \frac{X_{out}(s)}{X_{in}(s)}$

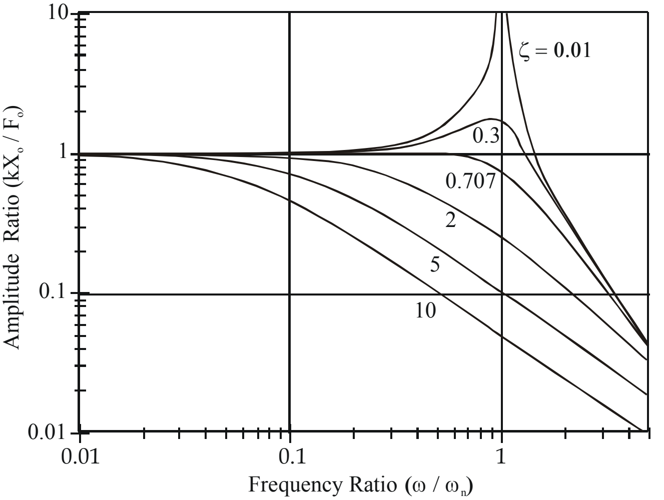
*amplitude ratio:* $\frac{A_{out}}{A_{in}} = \text{mag}(G(j\omega)) = \|G(j\omega)\|$

*phase angle:* $\phi = \arg(G(j\omega)) = \angle G(j\omega)$

second order system frequency response:

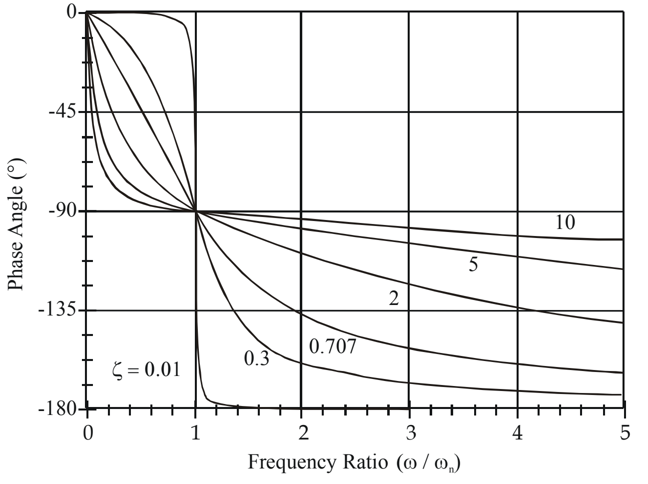
*amplitude ratio:*

$$\frac{X_o}{F_o/k} = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}^{1/2}}$$




*phase angle:*

$$\phi = -\tan^{-1}\left\{\frac{2\zeta}{\frac{\omega_n}{\omega} - \frac{\omega}{\omega_n}}\right\}$$



modeling analogies:

| Generic quantity | Mechanical translation | Mechanical rotation | Electrical | Hydraulic |
|---|---|---|---|---|
| Effort ($E$) | Force ($F$) | Torque ($T$) | Voltage ($V$) | Pressure ($P$) |
| Flow ($F$) | Speed ($v$) | Angular speed ($\omega$) | Current ($i$) | Volumetric flow rate ($Q$) |
| Displacement ($q$) | Displacement ($x$) | Angular displacement ($\theta$) | Charge ($q$) | Volume (V) |
| Momentum ($p$) | Linear momentum ($p = mv$) | Angular momentum ($h = J\omega$) | Flux linkage ($I = N\Phi = L\,i$) | Momentum/area ($\Gamma = IQ$) |
| Resistor ($R$) | Damper ($b$) | Rotary damper ($B$) | Resistor ($R$) | Resistor ($R$) |
| Capacitor ($C$) | Spring ($1/k$) | Torsion spring ($1/k$) | Capacitor ($C$) | Tank ($C$) |
| Inertia ($I$) | Mass ($m$) | Moment of inertia ($J$) | Inductor ($L$) | Inertance ($I$) |
| Inertia energy storage (special case) | $F = \dot{p}$ ($F = ma$) | $T = \dot{h}$ ($T = J\alpha$) | $V = \dot{\lambda}$ ($V = L\,di/dt$) | $P = \Gamma$ ($P = I\,dQ/dt$) |
| Capacitor energy storage | $F = k\,x$ | $T = k\theta$ | $V = (1/C)q$ | $P = (1/C)$ V |
| Dissipative | $F = b\,v$ | $T = B\omega$ | $V = Ri$ | $P = RQ$ |