

4.10.2 Frequency Response of a System

$$F_{\text{ext}}(t) = F_o \sin(\omega t) \quad (4.72)$$

$$x(t) = X_o \sin(\omega t + \phi) \quad (4.73)$$

Analytical Procedure to Determine the Frequency Response of a System

1. Find the Laplace transform of the system differential equation assuming initial conditions are zero: $x(0) = dx/dt(0) = 0$. The Laplace transform converts the differential equation into an algebraic equation that is related to the frequency response of the system.

$$\frac{1}{\omega_n^2} \frac{d^2 x(t)}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx(t)}{dt} + x(t) = \frac{1}{k} F_{\text{ext}}(t) \quad (4.74)$$

$$\left(\frac{1}{\omega_n^2} s^2 + \frac{2\zeta}{\omega_n} s + 1 \right) X(s) = \frac{1}{k} F_{\text{ext}}(s) \quad (4.75)$$

2. Find the **transfer function** of the system, which is the ratio of the output and input Laplace transforms.

$$G(s) = \frac{X(s)}{F_{\text{ext}}(s)} = \frac{1/k}{\left(\frac{1}{\omega_n^2} s^2 + \frac{2\zeta}{\omega_n} s + 1 \right)} \quad (4.76)$$

3. To simulate a harmonic input, replace s with $j\omega$ in the transfer function. This yields the frequency response behavior of the system.

$$G(j\omega) = \frac{1/k}{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right] + j \left(2\zeta \frac{\omega}{\omega_n} \right)} \quad (4.77)$$

4. Find the desired amplitude ratio between the output and input by determining the magnitude of the complex transfer function:

$$\text{mag}[G(j\omega)] = |G(j\omega)| \quad (4.78)$$

$$\frac{X_o}{F_o/k} = \frac{1}{\left\{ \left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]^2 + 4\zeta^2 \left(\frac{\omega}{\omega_n} \right)^2 \right\}^{1/2}} \quad (4.79)$$

5. Find the phase angle ϕ between the output and input by determining the argument of the complex transfer function:

$$\phi = \arg[G(j\omega)] = \angle G(j\omega) \quad (4.80)$$

$$\phi = 0 - \tan^{-1} \left\{ \frac{2\zeta \frac{\omega}{\omega_n}}{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]} \right\} = -\tan^{-1} \left\{ \frac{2\zeta}{\frac{\omega_n}{\omega} - \frac{\omega}{\omega_n}} \right\} \quad (4.81)$$